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Technical Report TR-63-3-BF

A Short Table of a Multiple Alternative Error Integral

by

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ABSTRACT

In this report are tabulated the values of

$$P_M(a, b) = \int_{-\infty}^{\infty} \phi(x)\, \Phi^{M-1}(ax+b)\, dx$$

for a = 1.0(0.1)1.4, b = 0(0.25)5, and M = 2, 3, 4, 5, 6, 7, 8, 9, 10, 16, 32, 64, 128, 256, 512. $\phi(x)$ is the normalized Gaussian probability density function and $\Phi(x)$ is the normal cumulative probability. The table was prepared by calculating $P_M(a, b)$ with an accuracy of approximately $\pm 5 \cdot 10^{-6}$, and rounding off to five places. Therefore an occasional error of one unit in the fifth place occurs.

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1. INTRODUCTION

The probability

$$P_M(a, b) = \int_{-\infty}^{\infty} \phi(x)\ \Phi^{M-1}(ax+b)\, dx$$

arises in at least two different modes of multiple alternative communication. $\phi(x)$ is the normalized Gaussian probability density function,

$$\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2),$$

and $\Phi(x)$ is the normal cumulative probability,

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)\, dy)$$

In the first,* one of M equiprobable equal energy equi-correlated signals or the null signal is transmitted. At the receiver, the largest of the sampled outputs of M phase-coherent matched filters is compared with a threshold. For the null signal transmitted, the probability that the threshold is exceeded is called the false detection probability, P_F, and is given by**

$$P_F(\lambda, \Gamma) = 1 - P_{M+1}\left(\sqrt{\frac{\lambda}{1-\lambda}}\ ,\ \frac{\Gamma}{\sqrt{1-\lambda}}\right)$$

where λ is the (common) correlation coefficient of the signal set and Γ is a normalized threshold.

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*A. H. Nuttall, "Error Probabilities for Equi-Correlated M-ary Signals Under Phase-Coherent and Phase-Incoherent Reception", IRE Trans. on Info. Th., Vol. IT-8, No. 4, pp. 305-314; July 1962. (See eq. (73).) Also, Technical Report TR-61-1-BF, Litton Systems, Inc., Waltham, Mass.; June 15, 1961. (See Section 5.)

**A very small table (240 values) of $P_F(\lambda, \Gamma)$ is available as Appendix D of TR-61-1-BF, through the use of eq. (5.35). The present table complements and greatly supplements the previous tabulation.

The second mode* of multiple alternative communication is phase-coherent, through (fast) Rayleigh fading. One of M equiprobable equal energy orthogonal signals is transmitted; at the receiver, the outputs of M radiometer-type** filters are sampled and a decision made in favor of the largest filter output. The probability of correct decision is approximately given by $P_M(a, b)$ through appropriate identification of a and b.

In general, the probability $P_M(a, b)$ furnishes the answer to the following statistical problem (which is not tied to any particular application). Consider a set $\{x_k\}$ of M independent Gaussian random variables with

$$\overline{x_1} = m \, ; \; \overline{x_k} = 0, \quad k \geq 2$$

and

$$\sigma^2(x_1) = \sigma_1^2 \, ; \; \sigma^2(x_k) = \sigma_2^2 \, , \quad k \geq 2.$$

Then the probability that the variable x_1 is larger than all the other variables is

$$\Pr(x_1 > x_2, \ldots, x_M)$$

$$= \int_{-\infty}^{\infty} dx_1 \, p_1(x_1) \int \cdots \int_{-\infty}^{x_1} dx_2 \ldots dx_M \, p_2(x_2) \ldots p_2(x_M)$$

$$= \int_{-\infty}^{\infty} dx_1 \frac{\exp\left[-\frac{(x_1 - m)^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\,\sigma_1} \left\{ \int_{-\infty}^{x_1} dx_2 \frac{\exp\left[-\frac{x_2^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\,\sigma_2} \right\}^{M-1}$$

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*A. H. Nuttall, "Linear Signal Processing Theory and Measurements", Technical Report TR-63-2-BF, Litton Systems, Inc., Waltham, Mass; April 30, 1963. (See Section 5.)

**R. Price and P. E. Green, Jr., "Signal Processing in Radar Astronomy--Communication by Fluctuating Multipath Media", Technical Report No. 234, Lincoln Laboratory, Mass. Inst. of Tech.; October 6, 1960.

$$= \int_{-\infty}^{\infty} \phi(x)\, \Phi^{M-1}\left(\frac{\sigma_1}{\sigma_2}\, x + \frac{m}{\sigma_2}\right) dx$$

$$= P_M\left(\frac{\sigma_1}{\sigma_2},\ \frac{m}{\sigma_2}\right)$$

Also, for completeness, we note that the probability that one of the other variables (not x_1) is largest is

$$\Pr(x_2 > x_1,\ x_3, \ldots, x_M) = \Pr(x_3 > x_1,\ x_2,\ x_4, \ldots, x_M) = \ldots$$

$$= \frac{1 - P_M\left(\frac{\sigma_1}{\sigma_2},\ \frac{m}{\sigma_2}\right)}{M-1}$$

2. SPECIAL CASES

For $a = 1$, $P_M(1, b)$ reduces to an integral already tabulated by Urbano.*

For $M = 2$, $P_2(a, b)$ can be readily integrated to yield

$$P_2(a, b) = \Phi\left(\frac{b}{\sqrt{1+a^2}}\right) .$$

For $b = 0$, $P_M(a, 0)$ may be integrated in closed form** for $M = 2, 3$, and 4. The results are

$$P_2(a, 0) = \frac{1}{2}$$

$$P_3(a, 0) = \frac{1}{2} - \frac{1}{\pi} \sin^{-1} \left[\frac{1}{2(1+a^2)}\right]^{1/2}$$

$$P_4(a, 0) = \frac{1}{2} - \frac{3}{2\pi} \sin^{-1} \left[\frac{1}{2(1+a^2)}\right]^{1/2}$$

And for $a = 1$, $b = 0$, we have, for all M,

$$P_M(1, 0) = 1/M$$

These special cases allow for numerous checks on the computations, which are tabulated in the following section.

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*R. H. Urbano, "Analysis and Tabulation of the M-Positions Experiment Integral and Related Error Function Integrals", AFCRC-TR-55-100, Electronics Research Directorate, Air Force Cambridge Research Center, Cambridge, Mass.; April, 1955.

**Nuttall, "Error Probabilities, etc.", TR-61-1-BF, loc. cit., eqs. (5.36)-(5.48).

a = 1.0

M \ b	0.0	0.25	0.50
2	.50000	.57015	.63816
3	.33333	.40633	.48259
4	.25000	.31835	.39331
5	.20000	.26297	.33453
6	.16667	.22470	.29252
7	.14285	.19659	.26082
8	.12500	.17501	.23595
9	.11111	.15789	.21584
10	.10000	.14396	.19922
16	.06250	.09509	.13861
32	.03125	.05121	.08012
64	.01563	.02738	.04571
128	.00781	.01456	.02581
256	.00391	.00771	.01445
512	.00195	.00407	.00804

M \ b	0.75	1.00	1.25
2	.70206	.76025	.81162
3	.55931	.63370	.70320
4	.47224	.55203	.62953
5	.41243	.49370	.57495
6	.36832	.44936	.53229
7	.33418	.41421	.49770
8	.30681	.38548	.46888
9	.28430	.36143	.44437
10	.26539	.34093	.42318
16	.19388	.26060	.33725
32	.11983	.17156	.23553
64	.07275	.11051	.16047
128	.04353	.06992	.10709
256	.02574	.04357	.07022
512	.01506	.02681	.04535

a = 1.0

M \ b	1.50	1.75	2.00
2	.85557	.89203	.92135
3	.76581	.82019	.86576
4	.70186	.76677	.82279
5	.65286	.72454	.78784
6	.61355	.68983	.75845
7	.58099	.66050	.73315
8	.55338	.63520	.71099
9	.52953	.61304	.69131
10	.50863	.59338	.67364
16	.42106	.50831	.59485
32	.31060	.39427	.48285
64	.22310	.29749	.38129
128	.15663	.21913	.29377
256	.10783	.15805	.22144
512	.07299	.11191	.16370

M \ b	2.25	2.50	2.75
2	.94419	.96145	.97408
3	.90261	.93136	.95303
4	.86931	.90650	.93513
5	.84151	.88525	.91950
6	.81765	.86665	.90558
7	.79675	.85010	.89302
8	.77818	.83519	.88155
9	.76147	.82162	.87100
10	.74629	.80917	.86122
16	.67667	.75044	.81392
32	.57199	.65730	.73496
64	.47082	.56157	.64892
128	.37819	.46865	.56051
256	.29713	.38266	.47413
512	.22882	.30624	.39326

a = 1.0

M \ b	3.00	3.25	3.50
2	.98305	.98922	.99333
3	.96879	.97988	.98741
4	.95637	.97156	.98203
5	.94531	.96402	.97708
6	.93530	.95710	.97248
7	.92615	.95069	.96817
8	.91769	.94472	.96411
9	.90984	.93911	.96027
10	.90248	.93382	.95662
16	.86607	.90700	.93773
32	.80228	.85786	.90160
64	.72875	.79810	.85538
128	.64896	.72974	.79976
256	.56676	.65561	.73639
512	.48578	.57887	.66753

M \ b	3.75	4.00	4.25
2	.99599	.99766	.99867
3	.99235	.99549	.99742
4	.98899	.99347	.99624
5	.98586	.99155	.99511
6	.98292	.98974	.99403
7	.98013	.98801	.99300
8	.97748	.98635	.99200
9	.97496	.98475	.99103
10	.97254	.98322	.99009
16	.95980	.97498	.98498
32	.93443	.95796	.97406
64	.90039	.93407	.95807
128	.85740	.90247	.93598
256	.80600	.86291	.90707
512	.74747	.81577	.87106

a = 1.0

M \ b	4.50	4.75	5.00
2	.99926	.99960	.99979
3	.99857	.99923	.99959
4	.99790	.99886	.99940
5	.99726	.99851	.99922
6	.99664	.99817	.99903
7	.99604	.99784	.99885
8	.99546	.99751	.99868
9	.99490	.99719	.99851
10	.99435	.99688	.99834
16	.99131	.99514	.99738
32	.98460	.99120	.99515
64	.97438	.98497	.99152
128	.95968	.97564	.98588
256	.93961	.96239	.97756
512	.91352	.94445	.96584

a = 1.1

M \ b	0.00	0.25	0.50
2	.50000	.56677	.63169
3	.34221	.41140	.48334
4	.26332	.32861	.39945
5	.21551	.27631	.34433
6	.18325	.23992	.30485
7	.15991	.21296	.27492
8	.14219	.19209	.25131
9	.12826	.17540	.23213
10	.11698	.16171	.21618
16	.07782	.11255	.15705
32	.04292	.06594	.09748
64	.02378	.03859	.06013
128	.01322	.02255	.03689
256	.00736	.01316	.02253
512	.00410	.00767	.01370

a = 1.1

b	0.75	1.00	1.25
M			
2	.69304	.74942	.79978
3	.55570	.62612	.69243
4	.47361	.54852	.62156
5	.41763	.49372	.56984
6	.37640	.45229	.52977
7	.34447	.41953	.49744
8	.31882	.39276	.47059
9	.29766	.37035	.44779
10	.27983	.35122	.42810
16	.21165	.27585	.34819
32	.13879	.19055	.25257
64	.09006	.12979	.18015
128	.05792	.08735	.12665
256	.03697	.05819	.08794
512	.02344	.03843	.06042

b	1.50	1.75	2.00
M			
2	.84351	.88043	.91074
3	.75287	.80621	.85177
4	.69033	.75285	.80777
5	.64326	.71153	.77276
6	.60592	.67802	.74379
7	.57523	.64999	.71913
8	.54934	.62598	.69773
9	.52706	.60506	.67884
10	.50760	.58658	.66198
16	.42633	.50728	.58771
32	.32369	.40174	.48374
64	.24115	.31179	.38997
128	.17675	.23773	.30861
256	.12773	.17848	.24025
512	.09117	.13218	.18432

a = 1.1

M \ b	2.25	2.50	2.75
2	.93492	.95368	.96783
3	.88948	.91970	.94317
4	.85438	.89262	.92295
5	.82576	.87003	.90573
6	.80161	.85061	.89067
7	.78073	.83358	.87728
8	.76236	.81840	.86521
9	.74597	.80472	.85422
10	.73118	.79226	.84411
16	.66439	.73457	.79625
32	.56623	.64572	.71911
64	.47271	.55647	.63758
128	.38732	.47080	.55541
256	.31203	.39162	.47590
512	.24757	.32076	.40155

M \ b	3.00	3.25	3.50
2	.97820	.98560	.99072
3	.96084	.97373	.98284
4	.94623	.96350	.97591
5	.93353	.95446	.96969
6	.92227	.94633	.96402
7	.91212	.93891	.95878
8	.90287	.93208	.95392
9	.89437	.92574	.94937
10	.88648	.91982	.94509
16	.84831	.89055	.92349
32	.78408	.83922	.88411
64	.71275	.77945	.83613
128	.63740	.71336	.78066
256	.56112	.64344	.71941
512	.48666	.57225	.65441

a = 1.1

M \ b	3.75	4.00	4.25
2	.99417	.99643	.99787
3	.98909	.99325	.99594
4	.98455	.99036	.99415
5	.98040	.98769	.99249
6	.97658	.98520	.99092
7	.97301	.98285	.98943
8	.96967	.98064	.98801
9	.96652	.97854	.98666
10	.96354	.97653	.98536
16	.94821	.96605	.97846
32	.91919	.94551	.96448
64	.88226	.91826	.94518
128	.83771	.88399	.91994
256	.78641	.84291	.88847
512	.72974	.79570	.85089

M \ b	4.50	4.75	5.00
2	.99876	.99930	.99961
3	.99762	.99864	.99925
4	.99655	.99802	.99890
5	.99555	.99743	.99856
6	.99459	.99687	.99824
7	.99367	.99633	.99793
8	.99280	.99580	.99763
9	.99196	.99530	.99733
10	.99114	.99481	.99705
16	.98676	.99212	.99546
32	.97761	.98636	.99197
64	.96450	.97780	.98660
128	.94668	.96575	.97878
256	.92362	.94956	.96790
512	.89502	.92875	.95341

a = 1.2

M \ b	0.00	0.25	0.50
2	.50000	.56357	.62555
3	.35047	.41610	.48407
4	.27570	.33805	.40509
5	.23002	.28860	.35328
6	.19887	.25399	.31609
7	.17612	.22818	.28780
8	.15868	.20805	.26539
9	.14482	.19185	.24710
10	.13352	.17847	.23182
16	.09335	.12956	.17441
32	.05560	.08113	.11458
64	.03337	.05094	.07515
128	.02013	.03202	.04919
256	.01218	.02014	.03212
512	.00739	.01267	.02093

M \ b	0.75	1.00	1.25
2	.68443	.73897	.78821
3	.55241	.61913	.68238
4	.47491	.54538	.61432
5	.42237	.49380	.56529
6	.38374	.45499	.52758
7	.35378	.42435	.49728
8	.32969	.39932	.47217
9	.30976	.37836	.45089
10	.29292	.36046	.43253
16	.22796	.28958	.35793
32	.15678	.20802	.26791
64	.10724	.14814	.19827
128	.07298	.10469	.14529
256	.04944	.07348	.10554
512	.03335	.05127	.07609

a = 1.2

M \ b	1.50	1.75	2.00
2	.83154	.86871	.89979
3	.74063	.79273	.83801
4	.67970	.73981	.79341
5	.63456	.69956	.75863
6	.59911	.66731	.73024
7	.57014	.64054	.70632
8	.54582	.61776	.68570
9	.52495	.59800	.66762
10	.50676	.58060	.65155
16	.43104	.50645	.58147
32	.33530	.40834	.48459
64	.25737	.32441	.39758
128	.19526	.25439	.32167
256	.14663	.19721	.25704
512	.10912	.15134	.20314

M \ b	2.25	2.50	2.75
2	.92512	.94525	.96084
3	.87624	.90761	.93264
4	.83976	.87865	.91031
5	.81065	.85502	.89172
6	.78645	.83506	.87576
7	.76576	.81775	.86176
8	.74772	.80249	.84927
9	.73174	.78884	.83799
10	.71741	.77649	.82770
16	.65349	.72020	.77984
32	.56126	.63556	.70493
64	.47442	.55211	.62771
128	.39527	.47271	.55108
256	.32507	.39940	.47748
512	.26423	.33342	.40872

a = 1.2

M \ b	3.00	3.25	3.50
2	.97260	.98126	.98747
3	.95204	.96666	.97738
4	.93532	.95451	.96880
5	.92114	.94402	.96127
6	.90879	.93475	.95453
7	.89781	.92642	.94840
8	.88792	.91884	.94278
9	.87891	.91188	.93758
10	.87063	.90543	.93273
16	.83132	.87425	.90882
32	.76739	.82160	.86698
64	.69855	.76248	.81805
128	.62739	.69887	.76333
256	.55634	.63291	.70442
512	.48747	.56661	.64307

M \ b	3.75	4.00	4.25
2	.99181	.99477	.99674
3	.98502	.99033	.99390
4	.97913	.98640	.99135
5	.97389	.98285	.98902
6	.96913	.97960	.98687
7	.96478	.97659	.98485
8	.96074	.97378	.98296
9	.95697	.97114	.98117
10	.95344	.96865	.97947
16	.93572	.95595	.97067
32	.90363	.93219	.95368
64	.86460	.90218	.93142
128	.81927	.86601	.90362
256	.76866	.82417	.87034
512	.71407	.77748	.83192

a = 1.2

b	4.50	4.75	5.00
M			
2	.99801	.99882	.99931
3	.99625	.99775	.99868
4	.99464	.99676	.99809
5	.99316	.99584	.99754
6	.99177	.99498	.99701
7	.99046	.99415	.99651
8	.98923	.99337	.99603
9	.98805	.99262	.99557
10	.98693	.99190	.99512
16	.98102	.98807	.99271
32	.96928	.98023	.98765
64	.95336	.96924	.98033
128	.93278	.95455	.97023
256	.90729	.93576	.95688
512	.87688	.91260	.93991

a = 1.3

b	0.00	0.25	0.50
M			
2	.50000	.56057	.61976
3	.35811	.42044	.48477
4	.28717	.34672	.41025
5	.24352	.29989	.36144
6	.21351	.26695	.32633
7	.19140	.24225	.29955
8	.17432	.22288	.27826
9	.16065	.20720	.26082
10	.14942	.19418	.24619
16	.10875	.14588	.19065
32	.06889	.09638	.13115
64	.04405	.06401	.09035
128	.02834	.04263	.06225
256	.01831	.02845	.04287
512	.01187	.01901	.02951

a = 1.3

M \ b	0.75	1.00	1.25
2	.67626	.72897	.77701
3	.54942	.61271	.67306
4	.47613	.54257	.60775
5	.42670	.49393	.56123
6	.39039	.45746	.52566
7	.36222	.42871	.49719
8	.33952	.40524	.47365
9	.32071	.38558	.45371
10	.30479	.36877	.43652
16	.24290	.30196	.36665
32	.17369	.22404	.28171
64	.12396	.16543	.21489
128	.08826	.12158	.16284
256	.06268	.08896	.12266
512	.04441	.06484	.09190

M \ b	1.50	1.75	2.00
2	.81979	.85701	.88865
3	.72913	.77988	.82463
4	.66994	.72767	.77980
5	.62669	.68859	.74547
6	.59300	.65759	.71777
7	.56563	.63205	.69463
8	.54273	.61042	.67481
9	.52313	.59173	.65752
10	.50607	.57533	.64222
16	.43526	.50577	.57600
32	.34563	.41419	.48539
64	.27195	.33560	.40428
128	.21221	.26933	.33320
256	.16439	.21432	.27205
512	.12651	.16925	.22026

$a = 1.3$

M \ b	2.25	2.50	2.75
2	.91494	.93628	.95320
3	.86310	.89531	.92162
4	.82562	.86481	.89744
5	.79630	.84045	.87775
6	.77224	.82015	.86111
7	.75188	.80276	.84667
8	.73426	.78754	.83392
9	.71874	.77403	.82249
10	.70489	.76187	.81214
16	.64378	.70719	.76469
32	.55696	.62660	.69224
64	.47596	.54835	.61906
128	.40225	.47442	.54736
256	.33656	.40621	.47890
512	.27906	.34453	.41497

M \ b	3.00	3.25	3.50
2	.96631	.97623	.98357
3	.94255	.95879	.97108
4	.92389	.94476	.96080
5	.90841	.93292	.95199
6	.89514	.92264	.94424
7	.88350	.91352	.93730
8	.87312	.90531	.93099
9	.86375	.89784	.92521
10	.85519	.89098	.91987
16	.81526	.85838	.89405
32	.75215	.80511	.85048
64	.68591	.74708	.80124
128	.61864	.68603	.74765
256	.55223	.62373	.69115
512	.48820	.56177	.63318

a = 1.3

M \ b	3.75	4.00	4.25
2	.98888	.99263	.99522
3	.98014	.98667	.99124
4	.97281	.98156	.98778
5	.96642	.97706	.98469
6	.96074	.97301	.98188
7	.95560	.96931	.97929
8	.95090	.96590	.97689
9	.94656	.96273	.97464
10	.94251	.95976	.97252
16	.92268	.94496	.96180
32	.88812	.91840	.94200
64	.84768	.88623	.91721
128	.80216	.84881	.88744
256	.75260	.80677	.85296
512	.70020	.76097	.81426

M \ b	4.50	4.75	5.00
2	.99696	.99811	.99885
3	.99437	.99647	.99783
4	.99209	.99499	.99690
5	.99002	.99365	.99605
6	.98812	.99240	.99525
7	.98636	.99123	.99450
8	.98471	.99013	.99379
9	.98316	.98909	.99311
10	.98169	.98810	.99246
16	.97416	.98296	.98905
32	.95984	.97292	.98222
64	.94134	.95953	.97283
128	.91841	.94244	.96049
256	.89105	.92143	.94488
512	.85943	.89645	.92580

a = 1.4

M \ b	0.00	0.25	0.50	0.75
2	.50000	.55776	.61433	.66855
3	.36518	.42445	.48543	.54670
4	.29777	.35468	.41497	.47727
5	.25604	.31026	.36887	.43064
6	.22716	.27888	.33567	.39642
7	.20573	.25524	.31027	.36986
8	.18906	.23661	.29003	.34843
9	.17565	.22147	.27338	.33064
10	.16456	.20884	.25939	.31555
16	.12379	.16142	.20575	.25655
32	.08245	.11143	.14701	.18948
64	.05549	.07744	.10541	.14002
128	.03760	.05405	.07571	.10342
256	.02560	.03782	.05443	.07632
512	.01749	.02652	.03914	.05625

M \ b	1.00	1.25	1.50	1.75
2	.71946	.76625	.80835	.84546
3	.60682	.66443	.71837	.76769
4	.54005	.60178	.66099	.71640
5	.49408	.55759	.61957	.67855
6	.45972	.52397	.58753	.64879
7	.43267	.49715	.56162	.62441
8	.41059	.47500	.54000	.60386
9	.39208	.45627	.52154	.58615
10	.37626	.44012	.50550	.57065
16	.31315	.37445	.43904	.50522
32	.23870	.29415	.35484	.41940
64	.18160	.23010	.28507	.34556
128	.13781	.17925	.22771	.28275
256	.10431	.13907	.18095	.22992
512	.07877	.10751	.14311	.18591